AN ELEMENTARY, ILLUSTRATIVE PROOF OF THE RADO-HORN THEOREM

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Abstract. The Rado-Horn theorem provides necessary and sufficient conditions for when a collection of vectors can be partitioned into a fixed number of linearly independent sets. Such partitions exist if and only if every subset of the vectors satisfies the so-called Rado-Horn inequality. Today there are at least six proofs of the Rado-Horn theorem, but these tend to be extremely delicate or require intimate knowledge of matroid theory. In this paper we provide an elementary proof of the Rado-Horn theorem as well as elementary proofs for several generalizations including results for the redundant case when the hypotheses of the Rado-Horn theorem fail. Another problem with the existing proofs of the Rado-Horn Theorem is that they give no information about how to actually partition the vectors. We start by considering a specific partition of the vectors, and the proof consists of showing that this is an optimal partition. We further show how certain structures we construct in the proof are at the heart of the Rado-Horn theorem by characterizing subsets of vectors which maximize the Rado-Horn inequality. Lastly, we demonsrate how these results may be used to select an optimal partition with respect to spanning properties of the vectors.

1. Introduction

The terminology *Rado-Horn theorem* was first introduced in [3]. This theorem [10, 13] provides necessary and sufficient conditions for a collection of vectors to be partitioned into k linearly independent sets:

Theorem 1.1. *(Rado-Horn) Consider the vectors $\Phi = \{\varphi_i\}_{i=1}^M$ in a vector space. Then the follwing are equivalent.*

(i) The set Φ can be partitioned into sets $\{A_i\}_{i=1}^k$ such that A_i is a linearly independent set for all $i = 1, 2, \ldots, k$.
(ii) For any subset $J \subseteq \Phi$, we have $|J| \,/ \dim \operatorname{span}(J) \leq k$.

The Rado-Horn theorem has found application in several areas including progress on the Feichtinger conjecture [5], a characterization

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of Sidon sets in $\Pi_{k=1}^{\infty}\mathbb{Z}_p$ [11, 12], and a notion of redundancy for finite frames [1]. A generalized version of the Rado-Horn theorem has found use in frame theory as well where redundancy is at the heart of the subject [2].

Unfortunatly, proving the Rado-Horn theorem tends to very intricate or complicated. Pisier, when discussing a characterization of Sidon sets in $\Pi_{k=1}^{\infty}\mathbb{Z}_p$ states "... d'un lemme d'alébre dû à Rado-Horn dont la démonstration est relativement délicate" [12]. Today there are at least six proofs of the Rado-Horn theorem [4, 5, 8, 9, 10, 13]. The theorem was proved in a more general algebraic setting in [10, 13] and then for matroids in [8]. Each of these proofs are extremely delicate. Harary and Welsh [9] improved upon the matroid version of the Rado-Horn theorem with a short and elegant proof; however, their argument requires a development of certain deep structures within matroid theory. The Rado-Horn theorem was generalized in [4] to include partitions of a collection of vectors with subsets of specified sizes removed, and the authors also proved results for the redundant case - the case where a collection of vectors cannot be partitioned into k linear independent sets. Unfortuneately the proofs for these refinements to the theorem are even more delicate than the original. Finally, the Rado-Horn theorem was rediscovered in [5], where the authors give an induction proof which may be considered elementary. However, the proof has some limitations as it does not clearly generalize or describe the redundant case; it does not reveal the origin of the Rado-Horn inequality.

In this paper, we present a elementary proof which is at the core of the Rado-Horn theorem. With slight modification, these simple arguments prove a generalization of the Rado-Horn theorem and provide results for the redundant case similar to those in [4]. Perhaps most significanlty, the arguements we present may be thought of visually and provide insight into the specific conditions which give rise to the inequality in the Rado-Horn theorem.

This paper is organized into three sections. The first develops constructions and main arguments used througout the paper. The second section uses these tools to prove the Rado-Horn theorem and its generalization. The final section describes which subsets maximise the Rado-Horn inequality and how this may be used to concretely construct a so-called fundamental partition.

2. Preliminary Results

Given a set of vectors, $\Phi = \{\varphi_i\}_{i=1}^{M}$, a main construction for our proofs will be a partition $F = \{F_i\}_{i=1}^{\ell}$ of Φ which is optimal in the

sense that we have as many spanning sets as possible. Then given this number of spanning sets, we have the partition also with the maximum number of sets spanning codimension one as possible. This property continues through the entire partition of vectors. We give a more formal definition below.

Definition 2.1. *Given vectors $\Phi = \{\varphi_i\}_{i=1}^M$, we call a partition $\{A_i\}_{i=1}^k$ of Φ an* **ordered partition** *if $|A_i| \geq |A_{i+1}|$ for all $i = 1, \ldots, k-1$.*

Definition 2.2. *Given vectors $\Phi = \{\varphi_i\}_{i=1}^M$, let $\{P_i\}_{i=1}^m$ be all possible ordered partitions of Φ into linearly independent sets. Let A_{ij} denote the jth set in the ith partition so that $P_i = \{A_{ij}\}_{i=1,j=1}^{m,r_i}$. Now define*

$$a_1 = \sup_{i=1,\ldots,m} |A_{i1}|.$$

Then consider only the partitions $\{P_i : |A_{i1}| = a_1\}$ and define

$$a_2 = \sup_{\{i:|A_{i1}|=a_1\}} |A_{i2}|.$$

We continue to consider fewer and fewer partitions so that given $a_1, \ldots, a_n$,

$$a_{n+1} = \sup_{\{i:|A_{i1}|=a_1,\ldots,A_{in}=a_n\}} \left|A_{i(n+1)}\right|.$$

When $\sum_{i=1}^{\ell} a_i = M$, any remaining partition is in the set $\{P_i : |A_{i1}| = a_1, \ldots, |A_{i\ell}| = a_\ell\}$. We call any such ordered partition of Φ into linearly indpendent sets $F = \{F_i\}_{i=1}^{\ell}$ a **fundamental partition**.

We define a fundamental partition in this manner simply because this definition makes existence clear. However, the following theorem gives an equivalent definition and is Theorem 1 from [7].

Theorem 2.3. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a collection of vectors. Then $F = \{F_i\}_{i=1}^{\ell}$ is a fundamental partition if and only if for any other ordered partition $\{P_i\}_{i=1}^k$ of Φ into linearly independent sets,*

(i) $\ell \leq k$
(ii) $\sum_{i=1}^{j} |P_i| \leq \sum_{i=1}^{j} |F_i|, j = 1, 2, \ldots, \ell.$

That is, an ordered partition of Φ is a fundamental partition if and only if it *majorizes* every other ordered partition of Φ into linearly independent sets.

It is helpful to view a fundamental partition as a Young diagram where each square represents a vector, and the rows correspond to the sets of the partition. See figure 1.

We will occasionally need to move vectors within a fundamental partition. If we do this carefully, the linear independence properties and

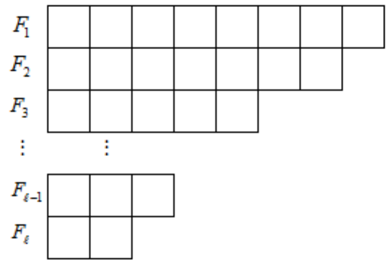

Figure 1. Example of a fundamental partition

the subspaces spanned by many of the sets in the partition will remain unchanged.

Proposition 2.4. *Let $\Phi = \{\varphi_i\}_{i=1}^M$ be a collection of linearly independent vectors. Suppose $\psi \in \operatorname{span}(\Phi)$ so that $\psi = \sum_{i=1}^M c_i\varphi_i$. Then for any $j \in \{1, \ldots, M\}$ such that $c_j \neq 0$, $\Psi_j = \{\Phi, \psi\} \setminus \{\varphi_j\}$ is linearly independent and $\operatorname{span}(\Psi_j) = \operatorname{span}(\Phi)$.*

Proof. Suppose for some scalars a_i,

$$\begin{aligned}\sum_{i=1, i\neq j}^M a_i\varphi_i + a_j\psi &= \sum_{i=1, i\neq j}^M a_i\varphi_i + a_j \sum_{i=1}^M c_i\varphi_i \\ &= \sum_{i=1, i\neq j}^M (a_i + a_jc_i)\varphi_i + a_jc_j\varphi_j \\ &= 0.\end{aligned}$$

Since $c_j \neq 0$, we must have $a_j = 0$ by linear independence of Φ, but then $a_i = 0$ for all $i \in \{1, \ldots j-1, j+1, \ldots, M\}$. Thus Ψ_j is linearly independent.

Since $\psi \in \operatorname{span}(\Phi)$, then $\operatorname{span}(\Psi_j) \subseteq \operatorname{span}(\Phi)$. It follows $\operatorname{span}(\Psi_j) = \operatorname{span}(\Phi)$ since $|\Psi_j| = |\Phi|$. $\square$

We will also be interested in which vectors are contained in the spans of each set in a fundamental partition. The following lemma is trivial but does provides some information in this regard.

Lemma 2.5. *Let $\{F_i\}_{i=1}^\ell$ be a fundamental partition of $\Phi = \{\varphi_i\}_{i=1}^M$. Then $\operatorname{span}(F_j) \subseteq \operatorname{span}(F_i)$ for $i \leq j$.*

Proof. Suppose there existed some $\varphi \in F_j$, such that $\varphi \notin \operatorname{span}(F_i)$. Then $\{F_i, \varphi\}$ would be linearly independent contradicting our assumption that F is a fundamental partition. $\square$

This shows that any vector is contained in the spans of the sets before it in the partition; however, we may carefully choose certain vectors which must be contained in the spans of all sets, with the possible exception of F_ℓ.

Lemma 2.6. *Let $\{F_i\}_{i=1}^{\ell}$ be a fundamental partition of $\Phi = \{\varphi_i\}_{i=1}^{M}$. Pick any $\varphi_\ell \in F_\ell$ and let $S_{\ell-1} \subseteq F_{\ell-1}$ be smallest set such that $\varphi \in$ $\operatorname{span}(S_{\ell-1})$. Fix any $j \leq \ell - 1$, and let S_j be the smallest subset of F_j such that $\operatorname{span}(S_{\ell-1}) \subseteq \operatorname{span}(S_j)$. Then $\operatorname{span}(S_j) \subseteq \operatorname{span}(F_i)$, $i = 1, \ldots, \ell - 1$.*

Proof. Clearly the sets $S_{\ell-1}$ and S_j exist by Lemma 2.5. We will prove the statement for $i = \ell - 1$. The result will then follow for all $i = 1, \ldots, \ell - 1$ since $\operatorname{span}(F_{\ell-1}) \subseteq \operatorname{span}(F_i)$ for $i \leq \ell - 1$.

We will assume the result fails and get a contradiction. So assume there exists some $\varphi_j \in S_j$ such that $\varphi_j \notin \operatorname{span}(F_{\ell-1})$. By Proposition 2.4, there exits some $\varphi_{\ell-1} \in S_{\ell-1}$ such that $\{S_j, \varphi_{\ell-1}\} \setminus \{\varphi_j\}$ is linearly independent with the same span as S_j. Similarly, $\{S_{\ell-1}, \varphi_\ell\} \setminus \{\varphi_{\ell-1}\}$ is linearly independent and has the same span as $S_{\ell-1}$. Thus we can partition $\Phi \setminus \{\varphi_j\}$ into ℓ linearly independent sets, say $G = \{G_i\}_{i=1}^{\ell}$ given by

$$G_i = \begin{cases} \{F_j, \varphi_{\ell-1}\} \setminus \{\varphi_j\} & \text{for } i = j \\ \{F_{\ell-1}, \varphi_\ell\} \setminus \{\varphi_{\ell-1}\} & \text{for } i = \ell - 1 \\ F_\ell \setminus \{\varphi_\ell\} & \text{for } i = \ell \\ F_i & \text{for } i \neq j, \ell - 1, \ell \end{cases}$$

Notice $|G_i| = |F_i|$ and $\operatorname{span}(G_i) = \operatorname{span}(F_i)$ for $i = 1, \ldots, \ell - 1$. Then $\{G_{\ell-1}, \varphi_j\}$ is linearly independent with $|\{G_{\ell-1}, \varphi_j\}| > |F_{\ell-1}|$ contradicting the fact that F was a fundamental partition. $\square$

Corollary 2.7. *Let $\{F_i\}_{i=1}^{\ell}$ be a fundamental partition of $\Phi = \{\varphi_i\}_{i=1}^{M}$. Pick any $\varphi_\ell \in F_\ell$ and let $S_{\ell-1}^{(1)} \subseteq F_{\ell-1}$ be smallest set such that $\varphi_\ell \in$ $\operatorname{span}(S_{\ell-1})$. Let $S_i^{(1)} \subseteq F_i$, $i = 1, \ldots \ell - 1$ be the smallest subset such that $\operatorname{span}(S_{\ell-1}^{(1)}) \subseteq \operatorname{span}(S_i^{(1)})$. Pick a $S_{j_1}^{(1)}$ such that $\left|S_{j_1}^{(1)}\right| \geq \left|S_i^{(1)}\right|$ for all $i = 1, \ldots, \ell - 1$, and set $S_{j_1}^{(1)} = S_{j_1}^{(2)}$. Now define $S_i^{(2)} \subseteq F_i$, $i = 1, \ldots \ell - 1$ as the smallest subset such that $\operatorname{span}(S_{j_1}^{(2)}) \subseteq \operatorname{span}(S_i^{(2)})$ and choose $S_{j_2}^{(2)}$ so that $\left|S_{j_2}^{(2)}\right| \geq \left|S_i^{(2)}\right|$ for all $i = 1, 2 \ldots, \ell - 1$. Continue this process so that $S_i^{(n)} \subseteq F_i$, $i = 1, \ldots \ell - 1$ is the smallest subset such that $\operatorname{span}(S_{j_{n-1}}^{(n)}) \subseteq \operatorname{span}(S_i^{(n)})$. Then $\operatorname{span}(S_{j_n}^{(n)}) \subseteq \operatorname{span}(F_i)$, for $i = 1, \ldots, \ell - 1$.*

Proof. For $n = 1$, this is Lemma 2.6. Notice this gaurantees the sets $S_i^{(2)}$, $i = 1, \ldots, \ell - 1$ are well defined. It suffices to show $\mathrm{span}(S_{j_n}^{(n)}) \subseteq \mathrm{span}(F_{\ell-1})$. Then $S_{j_n}^{(n+1)} = S_{j_n}^{(n)}$, and $S_i^{(n+1)}$, $i = 1, \ldots, \ell - 1$ are well defined.

Suppose instead there existed some $\varphi_{j_n}^{(n)} \in S_{j_n}^{(n)}$ such that $\varphi_{j_n}^{(n)} \notin \mathrm{span}(F_{\ell-1})$. By Proposition 2.4, there exists some $\varphi_{j_{n-1}}^{(m_1)}$, $m_1 < n$, such that $\{S_{j_n}^{(n)}, \varphi_{j_{n-1}}^{(m_1)}\} \setminus \{\varphi_{j_n}^{(n)}\}$ is linearly independent and has the same span as $S_{j_n}^{(n)}$. There may be several such vectors $\varphi_{j_{n-1}}^{(m_1)}$, but we may choose a vector such that m_1 is minimal. Indeed simply note if $a < b$ and $j_a = j_b$ then $S_{j_a}^{(a)} \subseteq S_{j_b}^{(b)}$.

Then we consider $S_{j_{m_1}}^{(m_1)}$ and again apply Proposition 2.4. There exists some $\varphi_{j_{m_1-1}}^{(m_2)}$, $m_2 < m_1$, such that $\{S_{j_{m_1}}^{(m_1)}, \varphi_{j_{m_1-1}}^{(m_2)}\} \setminus \{\varphi_{j_{n-1}}^{(m_1)}\}$ is linearly independent and has the same span as $S_{j_{m_1}}^{(m_1)}$. Choose the smallest such m_2 for $\varphi_{j_{m-1}}^{(m_2)}$.

By continuing this process $\{m_i\}_{i=1}^k$ is a decreasing sequence which terminates with $m_k = 1$. One final application of Proposition 2.4 implies $\{S_{\ell-1}^{(1)}, \varphi_\ell\} \setminus \{\varphi_{\ell-1}^{(1)}\}$ is linearly independent and has the same span as $S_{\ell-1}^{(1)}$.

Thus we can partition $\Phi \setminus \{\varphi_{j_n}^{(n)}\}$ into ℓ sets of linear independent vectors, say $G = \{G_i\}_{i=1}^\ell$ where $|G_i| = |F_i|$ and $\mathrm{span}(G_i) = \mathrm{span}(F_i)$ for $i = 1, \ldots, \ell - 1$. However, recalling $\varphi_{j_n}^{(n)} \notin \mathrm{span}(F_{\ell-1})$, $\{G_{\ell-1}, \varphi_{j_n}^{(n)}\}$ is also linearly independent contradicting that F was a fundamental partition.

□

The arguement in Corollary 2.7 is quite easy to visualize as shown in figure 2. Here we take an example where a fundamental partition contains six sets. The rows correspond to these sets F_i, and the $S_i^{(n)}$ are represented by the labeled vectors $\varphi_i^{(n)}$. When we have an appropriate vector that allows us to apply Proposition 2.4 (represented by a shaded square), we can move these vectors as indicated while maintaining linear independence of the row.

Using the above lemmas and corollaries, the Rado-Horn theorem will follow from the existence of so-called transversals in a fundamental partition. We borrow the term transversal from results in matroid theory where a fundamental partition is a basis for a sum of matroids [6, 7].

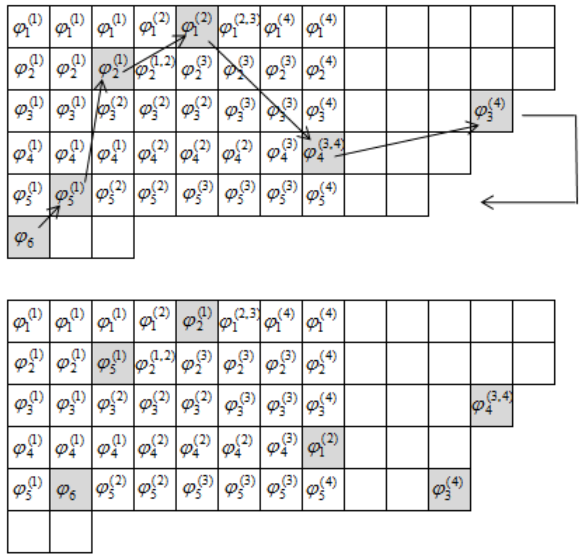

FIGURE 2. Original partition and resulting partition after performing argument in corollary 2.7

Definition 2.8. *Given a fundamental partition* $\{F_i\}_{i=1}^{\ell}$ *of* $\Phi = \{\varphi_i\}_{i=1}^{M}$ *and* $t \leq \ell$*, we call* $T \subseteq \Phi$ *a* **t-transversal** *if* $T = \{S_i\}_{i=1}^{t}$*,* $S_i \subseteq F_i$*, and* $\operatorname{span}(S_i) = \operatorname{span}(S_j)$ *for all* $i, j \in \{1, \ldots, t\}$*.*

We first show the existence of transversals in a fundamental partition.

Corollary 2.9. *Consider the vectors* $\Phi = \{\varphi_i\}_{i=1}^{M}$ *with a fundamental partition* $\{F_i\}_{i=1}^{\ell}$*. Fix* $t < \ell$ *and choose any* $\varphi_k \in F_k$ *where* $t < k$*. Then* $\{F_i\}_{i=1}^{t}$ *contains a* t*-transversal,* $T = \{S_i\}_{i=1}^{t}$*, with* $\varphi_k \in \operatorname{span}(S_i)$ *for all* $i \in \{1, \ldots, t\}$*.*

Proof. Notice if $F = \{F_i\}_{i=1}^{\ell}$ is a fundamental partition and we remove sets F_i, $i = t+1, \ldots, k-1, k+1, \ldots, \ell$, then $(\{F_i\}_{i=1}^{t}, F_k)$ remains a fundamental partition for its vectors. It therefore suffices to prove the statement for $t = \ell - 1$, and $k = \ell$.

Consider the sets $S_i^{(n)}$, $i = 1, \ldots, \ell - 1$, $n = 1, 2, \ldots$ as given in Corollary 2.7 where again $S_{j_n}^{(n)}$ is a largest such set for each n. Notice $\operatorname{span}(S_{j_n}^{(n)}) \subseteq \operatorname{span}(S_{j_{n+1}}^{(n+1)})$ for all $i = 1, \ldots, \ell - 1$. Since we have only finitely many vectors, there exits a n_0 such that

$$\left| S_{j_{n_0-1}}^{(n_0-1)} \right| = \left| S_{j_{n_0}}^{(n_0)} \right| .$$

Then

$$\left|S^{(n_0)}_{j_{n_0-1}}\right| = \left|S^{(n_0)}_i\right|.$$

Since $\text{span}(S^{(n_0)}_{j_{n_0-1}}) \subseteq \text{span}(S^{(n_0)}_i)$ for all $i = 1, \ldots, \ell-1$, we conclude

$$\text{span}(S^{(n_0)}_{j_{n_0-1}}) = \text{span}(S^{(n_0)}_i)$$

for all $i \in 1, \ldots, \ell-1$. Clearly $\varphi_\ell \in S^{(n_0)}_i$ and $S^{(n_0)}_i \subseteq F_i$ for all i by construction. Set $S_i = S^{(n_0)}_i$, and we have the desired $\ell - 1$ transversal. □

This argument shows there may be multiple such transversals in a fundamental partition. It is simple to see that given multiple t-transversals, there exists a t-transversal containing them.

Lemma 2.10. *Let $F = \{F_i\}_{i=1}^{\ell}$ be a fundamental partition of $\Phi = \{\varphi_i\}_{i=1}^M$. Suppose $T_1 = \{U_i\}_{i=1}^t$ and $T_2 = \{V_i\}_{i=1}^t$ are t-transversals in F. Then $T = \{U_i \cup V_i\}_{i=1}^t$ is a t-transversal.*

Proof. For any $i = 1, \ldots, t$, $U_i \cup V_i \subseteq F_i$ is linearly independent. We then have

$$\begin{aligned}\text{span}(U_i \cup V_i) &= \text{span}(U_i) + \text{span}(V_i)\\ &= \text{span}(U_j) + \text{span}(V_j)\\ &= \text{span}(U_j \cup V_j)\end{aligned}$$

for all $i, j \in \{1, \ldots, t\}$, and T is a t-transversal. □

We are now ready to prove the Rado-Horn theorem.

3. Proofs of Rado-Horn and its Generalizations

We begin with the original.

Theorem 3.1. *(Rado-Horn) Consider the vectors $\Phi = \{\varphi_i\}_{i=1}^M$. Then the follwing are equivalent.*

(i) The set Φ can be partitioned into sets $\{A_i\}_{i=1}^k$ such that A_i is a linearly independent set for all $i = 1, 2, \ldots, k$.
(ii) For any subset $J \subseteq \Phi$, we have $|J| / \dim \text{span}(J) \le k$.

Proof. ($i \Rightarrow ii$). This direction is essentially trivial. Suppose $\{A_i\}_{i=1}^k$ is a partition of Φ such that A_i is a linearly independent set for all $i = 1, 2, \ldots, k$. Then for any $J \subset \Phi$, let $J_i = J \cap A_i$, $i = 1, \ldots, k$. Then

$$|J| = \sum_{i=1}^k |J_i| = \sum_{i=1}^k \dim \text{span}(J_i) \le k \dim \text{span}(J)$$

giving the result.

($ii \Rightarrow i$). Suppose Φ cannot be partitioned into k linearly independent sets. Then for a fundamental partition $\{F_i\}_{i=1}^{\ell}$, we must have $\ell > k$. By Corollary 2.9, for any $\varphi \in F_\ell$, $\{F_i\}_{i=1}^{k}$ contains a k-transversal, T with $\varphi \in T$. Then we have

$$\frac{|T, \{\varphi\}|}{\dim \operatorname{span}(T, \{\varphi\})} = k + \frac{1}{\dim \operatorname{span}(T)} > k. \tag{1}$$

□

It is a simple matter to adapt the idea of this proof to show a generalized version of the Rado-Horn theorem. This theorem originally appeared in [4].

Theorem 3.2. *(Generalized Rado-Horn) Consider the vectors $\Phi = \{\varphi_i\}_{i=1}^{M}$. Then the follwing are equivalent.*

(i) There exists a subset $H \subseteq \Phi$ such that $\Phi \setminus H$ can be partitioned into k linearly independent sets.
(ii) For any subset $J \subseteq \Phi$, we have $(|J| - |H|)/\dim \operatorname{span}(J) \leq k$.

Proof. ($i \Rightarrow ii$). Suppose such a partition of $\Phi \setminus H$ exists. Then this direction remains trivial since by for any $J \subseteq \Phi$, we have

$$\frac{|J| - |H|}{\dim \operatorname{span}(J)} \leq \frac{|J \setminus H|}{\dim \operatorname{span}(J \setminus H)} \leq k$$

by the original Rado-Horn theorem.

($ii \Rightarrow i$). For the reverse direction, fix some L and suppose for every $H \subseteq \Phi$ with $|H| = L$, $\Phi \setminus H$ cannot be partitioned into k linearly independent sets. Take a fundamental partition $F = \{F_i\}_{i=1}^{\ell}$ of Φ. Then $\ell > k$ and $\sum_{i=k+1}^{\ell} |F_i| > L$ for otherwise such a partition would exist. Now consider $G = \{F_i\}_{i=1}^{k+1}$, and notice this is still a fundamental partition for a subset of Φ. Then Corollary 2.9 and Lemma 2.10 imply G contains a k-transversal $T = \{T_i\}_{i=1}^{k}$ such that $\operatorname{span}(F_{k+1}) \subseteq \operatorname{span}(T)$.

Now consider $J = \{T, F_{k+1}, \ldots, F_\ell\}$ and H as any subset of cadinality L. Then

$$\frac{|J| - |H|}{\dim \operatorname{span}(J)} > \frac{|T| + 1}{\dim \operatorname{span}(J)} = k + \frac{1}{\dim \operatorname{span}(J)} > k.$$

□

We next consider the redundant case. The following redundant versions of Rado-Horn were also originally proved in [4]. The transversals in a fundamental partition simply explain why the Rado-Horn inequality can fail when Φ cannot be partitioned into k linearly independent sets.

Theorem 3.3. *(Redundant Rado-Horn 1) Consider the vectors $\Phi = \{\varphi_i\}_{i=1}^M$ in a vector space V. If this set cannot be partitioned into k linearly independent sets, then there exists a partition $\{A_i\}_{i=1}^k$ of the Φ and a subspace S of V such that the following hold:*

(i) For all $1 \leq i \leq k$, there exists a subset $S_i \subseteq A_i$ such that $S = \mathrm{span}(S_i)$.
(ii) For $J = \{\varphi : \varphi \in S\}$, $|J| / \dim \mathrm{span}(J) > k$.

Proof. Take a fundamental partition $F = \{F_i\}_{i=1}^{\ell}$. Then for $k < \ell$ the hypothesis of Rado-Horn are not met. Choose any $\varphi \in F_\ell$, so there exists a k-transveral, T, in F which contains φ in its span. Simply consider the partition $\{A_i\}_{i=1}^k = (F_1, \dots, F_{k-1}, \{F_k, F_{k+1}, \dots, F_{\ell\}}$ and define the subspace $S = \mathrm{span}(T)$. Then (1) holds since T is a k-transversal, and (2) is clearly true since

$$\frac{|J|}{\dim \mathrm{span}(J)} \geq \frac{|T, \{\varphi\}|}{\dim \mathrm{span}(T, \{\varphi\})} > k$$

as in equation (1). □

There is one difference between this result when compared to the original. It is clear from the above arguement that the subspace S may not be unique. Indeed picking a different φ may lead to a different transversal and thus a different subspace. By taking several transversals and considering their union, we can obtain another result on the subspace S which is more akin to the original theorem [4].

Corollary 3.4. *(Redundant Rado-Horn 2) Consider the vectors $\Phi = \{\varphi_i\}_{i=1}^M$ in a vector space V. If this set cannot be partitioned into k linearly independent sets, then there exists a partition $\{A_i\}_{i=1}^k$ of Φ and a subspace S of V such that the following hold:*

(i) For all $1 \leq i \leq k$, there exists a subset $S_i \subseteq A_i$ such that $S = \mathrm{span}(S_i)$.
(ii) For $J = \{\varphi : \varphi \in S\}$, $|J| / \dim \mathrm{span}(J) > k$.
(iii) For all $1 \leq i \leq k$, $A_i \setminus S_i$ is linearly independent.

Proof. As before, take a fundamental partition $F = \{F_i\}_{i=1}^{\ell}$ of Φ, and consider the partition $\{A_i\}_{i=1}^k = (F_1, \dots, F_{k-1}, \{F_k, F_{k+1}, \dots, F_\ell\})$. We will show there exists a subspace S which satisfies i, ii, and iii for this partition.

By Corollary 2.9, for each $\varphi_i \in F_j$, $j = k+1, \dots, \ell$, there exits a k-transversal, say T_i, of F containing φ_i in $\mathrm{span}(T_i)$. By Lemma 2.10, the set

$$T = \cup_{\{i:\varphi_i \in F_j, j=k+1,\dots,l\}} T_i$$

is a k-transversal of F which satisfies $\varphi_i \in \operatorname{span}(T)$ for all $\varphi_i \in F_j$. Thus

$$\operatorname{span}(F_j) \subseteq \operatorname{span}(T)$$

for all $j = k+1, \dots, \ell$.

Finally, set $S = \operatorname{span}(T)$ with $S_i = T \cap F_i$ for $i = 1, \dots, k-1$ and $S_k = T \cap \{F_k, F_{k+1}, \dots, F_\ell\}$. Then i and ii follow as in Theorem 3.3 since T is a k-transversal which contains in its span at least one $\varphi \in F_j$, $j > k$ (in this case all of them). Clearly for $i = 1 \dots, k-1$, $A_i \setminus S_i \subseteq F_i$ is linearly independent. Lastly by the way we constructed our transversal,

$$\begin{aligned} A_k \setminus S_k &\subseteq \{F_k, F_{k+1}, \dots, F_l\} \setminus \{F_{k+1}, \dots, F_\ell\} \\ &\subseteq F_k \end{aligned}$$

which is also linearly independent. □

4. Constructing a Fundamental Partition

We have shown how the existence of transversals within a fundamental partition leads to an elementary proof of the Rado-Horn theorem. We will now exhibit how the Rado-Horn inequality may be used to find transversals and construct a fundamental partition. We begin by describing subsets which maximizes the Rado-Horn inequality.

Lemma 4.1. *Given vectors $\Phi = \{\varphi_i\}_{i=1}^M$ and a fundamental partition $F = \{F_i\}_{i=1}^\ell$, suppose $J \subset \Phi$ maximizes $|J|/\dim \operatorname{span}(J)$. Then J is comprised of an $(\ell-1)$-transversal in F together with the set $\{\varphi : \varphi \in F_\ell, \varphi \in \operatorname{span}(T)\}$.*

Proof. Suppose J maximized the Rado-Horn inequality but did not include such an $\ell-1$ transversal. Now J cannot be partitioned into fewer than ℓ linearly independent sets, so consider a fundamental partition $F' = \{F'_i\}_{i=1}^\ell$ of J. Then F' contains a maximal $\ell-1$ transversal T where $\operatorname{span}(F'_\ell) \subseteq \operatorname{span}(T)$. Define $T' = T \cup F'_\ell$. Note then

$$\frac{|T'|}{\dim \operatorname{span}(T')} > \ell - 1 > \frac{|J| - |T'|}{\dim \operatorname{span}(J) - \dim \operatorname{span}(T')}.$$

Then

$$\begin{aligned} |T'| \dim \operatorname{span}(J) &= |T'| \left[\dim \operatorname{span}(T') + (\dim \operatorname{span}(J) - \dim \operatorname{span}(T'))\right] \\ &> |T'| \dim \operatorname{span}(T') + (|J| - |T'|) \dim \operatorname{span}(T') \\ &= |J| \dim \operatorname{span}(T') \end{aligned}$$

so $|T'| / \dim \operatorname{span}(T') > |J| / \dim \operatorname{span}(J)$, a contradiction. □

The previous lemma provides a means for finding a transversal in some unknown fundamental partition. Next we show that given the proper transversal, if we project our vectors onto the orthogonal complement of the span of the transversal, the nonzero vectors retain their spanning and linearly indpendence properties with respect to each other. That is, after removing all vectors in the span of the transversal and projecting, the structure of a fundamental partition for the remaining vectors, albeit unknown, is unchanged. We formalize this claim.

Lemma 4.2. *Consider the vectors $\Phi = \{\varphi_i\}_{i=1}^M$ and fundamental partition $F = \{F_j\}_{j=1}^{\ell}$. For sake of notation, consider F as a partition of the subscripts $\{1, \ldots, M\}$. Let $t < \ell$ and suppose T is a t-transversal of F which satisfies $\operatorname{span}(\{\varphi_i\}_{i\in F_{t+1}}) \subseteq \operatorname{span}(\{\varphi_i\}_{i\in T})$. Let P_T be the orthogonal projection onto $\operatorname{span}(\{\varphi\}_{i\in T})$ and suppose $F_j' = \{i : i \in F_j, i \notin T\}$. Then $\{F_j'\}_{j=1}^t$ is a fundamental partition of $\{(I - P_T)\varphi_i\}_{i\notin T}$.*

Proof. First note $\{(I - P_T)\varphi_i\}_{i\in F_j'}$ is linearly independent for each $j = 1, \ldots, t$. Indeed suppose there exists scalars $\{\alpha_i\}_{i\in F_j'}$ such that $\sum_{i\in F_j'} \alpha_i (I - P_T)\varphi_i = 0$. Then $\sum_{i\in F_j'} \alpha_i\varphi_i \in \operatorname{span}(T) = \operatorname{span}(\{\varphi_i\}_{i\in F_j\setminus F_j'})$, but $\{\varphi_i\}_{i\in F_j}$ is linearly independent. Thus $\alpha_i = 0$ for all $i \in F_j'$.

Now suppose these independent sets do not form a fundamental partition. Then there exists some other partition of $\{i : i \notin T\}$, say $\{A_j\}_{j=1}^s$ such that $\{(I - P_T)\varphi_i\}_{i\in A_j}$ is linearly independent for all $j = 1, \ldots, s$ and there is some k such that $|A_k| > |F_k'|$ but $|A_i| = |F_i'|$ for all $i < k$. Setting $F_j' = \emptyset$ for any $t < j \le s$, it now suffices to show $\{\varphi_i\}_{i\in F_j\setminus F_j'\cup A_j}$ is linearly independent for $j = 1, \ldots, s$, for this would contradict that F was a fundamental partition.

For scalars α_i, consider $\sum_{i\in F_j\setminus F_j'\cup A_j} \alpha_i\varphi_i = 0$. Under the projection $I - P_T$, this becomes

$$\sum_{i\in F_j\setminus F_j'\cup A_j} \alpha_i (I - P_T)\varphi_i = \sum_{i\in A_j} \alpha_i (I - P_T)\varphi_i = 0,$$

and $\alpha_i = 0$ for $i \in A_j$. But then

$$\sum_{i\in F_j\setminus F_j'\cup A_j} \alpha_i\varphi_i = \sum_{i\in F_j\setminus F_j'} \alpha_i\varphi_i = 0,$$

and $\alpha_i = 0$ for all $i \in F_j \setminus F_j' \cup A_j$. $\square$

We now use the lemmas in this section to construct a fundamental partition.

Construction of a Fundamental Partition

Let $\Phi = \{\varphi_i\}_{i=1}^M = \{\varphi_{1i}\}_{i=1}^M$ be a collection of vectors (we've added the extra index in order to track an iterative process of projections). Chose T_1 as a subset of the indices $\{1, \ldots, M\}$ such that of all subsets of Φ, $J = \{\varphi_i\}_{i \in T_1}$ maximizes

$$\frac{|J|}{\dim \operatorname{span}(J)}. \tag{2}$$

Then by Lemma 4.1, $\{\varphi_i\}_{i \in T_1}$ comprises a transversal in some fundamental partition. Let

$$\begin{aligned} t_1 &= \dim \operatorname{span}(\{\varphi_i\}_{i \in T_1}) \\ k_1 &= \lceil |T_1| / t_1 \rceil \\ s_1 &= |T_1| - (k_1 - 1)t_1. \end{aligned}$$

Then we know exactly how this transversal appears in a fundamental partition. It is not difficult to see that we may partition T_1 as $\{T_{1j}\}_{j=1}^{k_1}$ where

(i) $|T_{1j}| = t_1$, $j = 1, \ldots, k_1 - 1$
(ii) $|T_{1j}| = s_1$, $j = k_1$
(iii) $\operatorname{span}(\{\varphi_i\}_{i \in T_{1n}}) = \operatorname{span}(\{\varphi_j\}_{j \in T_{1m}})$, $n, m \neq k_1$
(iv) $\operatorname{span}(\{\varphi_i\}_{i \in T_{1k_1}}) \subseteq \operatorname{span}(\{\varphi_i\}_{i \in T_{1j}})$, $j = 1, \ldots, k_1 - 1$

Let P_{T_1} be the orthogonal projection of Φ onto $\operatorname{span}(\{\varphi_{1i}\}_{i \in T_1})$. Define $\Phi_2 = \{(I - P_{T_1})\varphi_{1i}\}_{i \notin T_1} = \{\varphi_{2i}\}_{i \notin T_1}$. Now chose T_2 as a subset of the indices in $\{1, \ldots, M\} \setminus T_1$ such of all subsets of Φ_2, $J = \{\varphi_{2i}\}_{i \in T_2}$ maximizes (2). Then $\{\varphi_{2i}\}_{i \in T_2}$ comprises a transversal in a fundamental partition of Φ_2. Let

$$\begin{aligned} t_2 &= \dim \operatorname{span}(\{\varphi_{2i}\}_{i \in T_2}) \\ k_2 &= \lceil |T_2| / t_2 \rceil \\ s_2 &= |T_2| - (k_2 - 1)t_2, \end{aligned}$$

and we may partition T_2 as $\{T_{2i}\}_{i=1}^{k_2}$ where

(i) $|T_{2i}| = t_2$, $i = 1, \ldots, k_2 - 1$
(ii) $|T_{2i}| = s_2$, $i = k_2$
(iii) $\operatorname{span}(\{\varphi_j\}_{j \in T_{2n}}) = \operatorname{span}(\{\varphi_j\}_{j \in T_{2m}})$, $n, m \neq k_2$
(iv) $\operatorname{span}(\{\varphi_j\}_{j \in T_{2k_2}}) \subseteq \operatorname{span}(\{\varphi_j\}_{j \in T_{2i}})$, $i = 1, \ldots, k_2 - 1$.

We continue so that P_{T_j} is the orthogonal projection of Φ_j onto $\operatorname{span}(\{\varphi_{ji}\}_{i \in T_j})$. Define $\Phi_{j+1} = \{(I - P_{T_j})\varphi_{ji}\}_{i \notin T_1 \cup \ldots \cup T_j} = \{\varphi_{(j+1)i}\}_{i \notin T_1 \cup \ldots \cup T_j}$. Now choose T_{j+1} as a subset of the indices $\{1, \ldots, M\} \setminus \{T_1 \cup \ldots \cup T_j\}$ such that of all subsets of Φ_{j+1}, $J = \{\varphi_{(j+1)i}\}_{i \in T_{j+1}}$ maximizes (2).

Then $\{\varphi_{(j+1)i}\}_{i\in T_{j+1}}$ comprises a transversal in a fundamental partition of Φ_{j+1}. Letting

$$\begin{aligned} t_{j+1} &= \dim \operatorname{span}(\{\varphi_{(j+1)i}\}_{i\in T_{j+1}}) \\ k_{j+1} &= \lceil |T_{j+1}| / t_{j+1} \rceil \\ s_{j+1} &= |T_{j+1}| - (k_{j+1}-1)t_{j+1}. \end{aligned}$$

may partition T_{j+1} as $\{T_{(j+1)i}\}_{i=1}^{k_{j+1}}$

(i) $\left|T_{(j+1)i}\right| = t_{j+1}$, $i = 1, \ldots, k_{j+1}-1$
(ii) $\left|T_{(j+1)i}\right| = s_{j+1}$, $i = k_{j+1}$
(iii) $\operatorname{span}(\{\varphi_j\}_{j\in T_{(j+1)n}}) = \operatorname{span}(\{\varphi_j\}_{j\in T_{(j+1)m}})$, $n, m \neq k_{j+1}$
(iv) $\operatorname{span}(\{\varphi_j\}_{j\in T_{(j+1)k_{j+1}}}) \subseteq \operatorname{span}(\{\varphi_j\}_{j\in T_{(j+1)i}})$, $i = 1, \ldots, k_{j+1}-1$

Notice $k_j \geq k_{j+1}$. There is a small techinical issue here in that if $k_j = k_{j+1}$, the hypotheses of Lemma 4.2 are not met, and we require this Lemma to gaurantee this iterative process leads to a fundamental partition. If $k_j = k_{j+1}$, we may redefine T_j as the larger transversal $T'_j = T_j \cup T_{j+1}$. Then recalculate t_j and s_j (k_j will remain unchanged) and continue. The transversal becomes larger until at some point $k_j > k_{j+1}$. An exception is when $k_j = 0$, but this is not an issue since $k_j = 0$ only when $\Phi_j = \emptyset$.

Suppose r is such that $k_r \neq 0$ but $k_{r+1} = 0$. Finally, for $i > k_j$ adopt the convention $T_{ji} = \emptyset$.

Then $F = \{F_i\}_{i=1}^{k_1}$ where

$$F_i = \cup_{j=1,\ldots,r}\{\varphi_\ell\}_{\ell\in T_{ji}}$$

for $i = 1, \ldots, k_1$ is a fundamental partition of Φ.

The fact that piecing together the vectors from the transversals of projections yields a fundamental partition follows imediately from Lemma 4.2, where we showed such projections do not change the structure of a fundamental partition. See figure 3 for an example of a fundmental partition showing values t_i, k_i, s_i.

Remark 4.3. By constructing a fundamental partition we have essentially used the Rado-Horn inequality to described many of the spanning properties of the vectors. For example, using the notation from the above construction, a collection of vectors $\Phi = \{\varphi_i\}_{i=1}^M$ span a $\sum_{i=1}^r t_i$-dimensional space and can be partitioned into at most k_r spanning sets when $t_r = s_r$ and at most $k_r - 1$ spanning sets when $t_r \neq s_r$.

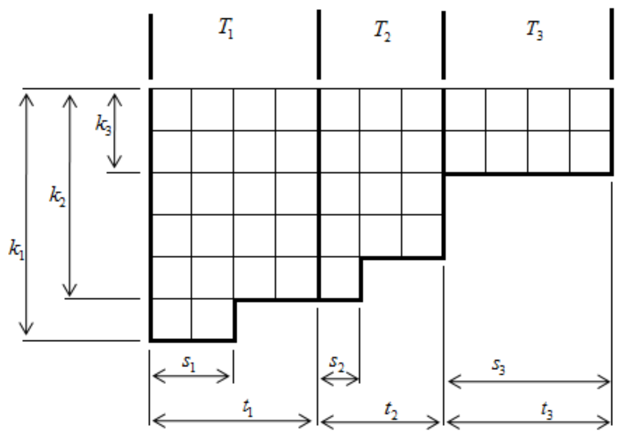

FIGURE 3. Fundamental partition constructed from transversals of appropriate projections

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